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Directed percolation near a wall

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Abstract. Series expansion methods are used to study directed bond percolation clusters on the square lattice whose lateral growth is restricted by a wall parallel to the growth direction. The percolation threshold p_c is found to be the same as that for the bulk. However, the values of the critical exponents for the percolation probability and mean cluster size are quite different from those for the bulk and are estimated by $\beta_1 = 0.7338 \pm 0.0001$ and $\gamma_1 = 1.8207 \pm 0.0004$ respectively. On the other hand the exponent $\Delta_1 = \beta_1 + \gamma_1$ characterizing the scale of the cluster size distribution is found to be unchanged by the presence of the wall.

The parallel connectedness length, which is the scale for the cluster length distribution, has an exponent which we estimate to be $\nu_{1\|} = 1.7337 \pm 0.0004$ and is also unchanged. The exponent τ_1 of the mean cluster length is related to β_1 and $\nu_{1\|}$ by the scaling relation $\nu_{1\|} = \beta_1 + \tau_1$ and using the above estimates yields $\tau_1 = 1$ to within the accuracy of our results. We conjecture that this value of τ_1 is exact and further support for the conjecture is provided by the direct series expansion estimate $\tau_1 = 1.0002 \pm 0.0003$.

Recently exact results have been obtained for directed compact clusters on the square lattice near a wall [1–3]. Such clusters are similar to ordinary percolation clusters except that they cannot branch and have no holes. These simplifying features allow several of the usual percolation functions to be derived analytically and the corresponding critical exponents have integer values. One of the main conclusions from these results was that although the moments of the cluster size and length distributions have exponents which change on introducing a wall parallel to the growth direction the exponents for the size and length scales remain the same. The other was that growth parallel to the wall is rather special in that any bias away from the wall results in bulk exponents. Similarly any bias towards the wall leads to wet wall exponents [1].

In this paper we find that the first of these conclusions extends to directed bond percolation. The exponents for directed percolation are not known exactly but numerical results show that, even in the absence of a wall, they are generally far from being integer and there is some doubt as to whether they even have rational values [4]. An interesting possibility raised by our results is that the mean cluster length in the presence of a wall parallel to the growth direction is an exceptional case and has the integer exponent $\tau_1 = 1$. Direct evidence for this value is provided by our analysis of the low density series expansion for the mean cluster length. Further support is provided by the scaling relation

$$\beta_1 + \tau_1 = \nu_{1\|} \qquad (1)$$

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together with series expansion estimates of β_1 and $\nu_{1\|}$. Here the subscript 1 on an exponent indicates its value in the presence of a wall. This relation is less well known than the one for the cluster size distribution, namely

$$\beta_1 + \gamma_1 = \Delta_1 \tag{2}$$

and is derived below. First we define the model and introduce some notation.

The directed square lattice may be described as having sites which are the points in the t–x plane with integer coordinates such that $t \geqslant 0$ and $t + x$ is even. There are two bonds leading from the general site (t, x) which terminate at the sites $(t + 1, x \pm 1)$. All bonds have probability p of being open to the passage of fluid and the source is placed at $(0, 0)$. This will be known as the bulk problem. A wall will be said to be present if the bonds leading to sites with $x < 0$ are always closed. The probability that fluid reaches column t but no further will be denoted by $r_t(p)$ and in this event the origin will be said to belong to a cluster of length t.

The percolation probability, the probability that the origin belongs to a cluster of infinite length, is defined by

$$P(p) = 1 - \sum_{t=0}^{\infty} r_t(p) = \sum_{t=0}^{\infty} (r_t(p_c) - r_t(p)) \sim (p - p_c)^{\beta} \qquad \text{for } p \to p_c^{+}. \tag{3}$$

If we suppose that the length distribution has the scaling form

$$r_t(p) \sim t^{-a} f(t/\xi_{\|}(p)) \tag{4}$$

then if

$$\xi_{\|}(p) \sim |p_c - p|^{-\nu_{\|}} \tag{5}$$

substitution in (3) yields

$$a = 1 + \frac{\beta}{\nu_{\|}}. \tag{6}$$

The mean cluster length is defined by

$$T(p) = \sum_{t=0}^{\infty} t r_t(p) \tag{7}$$

and using (4) we find that

$$T(p) \sim |p_c - p|^{-\tau} \tag{8}$$

where

$$\tau = \nu_{\|} - \beta. \tag{9}$$

The same argument holds in the presence of the surface and leads to (1). There is a close correspondence between the above derivation and that of (2) given in [5]. To obtain (2) it is only necessary to replace $r_t(p)$ by the cluster size distribution $p_s(p)$, $\xi_{\|}(p)$ by the scaling size $\sigma(p)$, which diverges with critical exponent Δ, and $T(p)$ by the mean cluster size $S(p)$ which diverges with exponent γ.

The mean size and the parallel and perpendicular scaling lengths are obtained from the pair connectedness function $C(t, x; p)$ which is the probablity that there is an open path from the origin to the site (t, x). The moments are defined by

$$\mu_{m,n}(p) = \sum_{\text{sites}} t^m x^n C(t, x; p) \tag{10}$$

in terms of which $S(p) = \mu_{00}(p)$. Assuming a scaling form for $C(t, x; p)$ similar to (4), where x is scaled by $\xi_\perp(p)$, it follows that

$$\xi_\parallel(p) \sim \frac{\mu_{m0}(p)}{\mu_{m-1,0}(p)} \qquad \text{and} \qquad \xi_\perp(p) \sim \frac{\mu_{0n}(p)}{\mu_{0,n-1}(p)}. \tag{11}$$

The series expansions are obtained by a transfer matrix method similar to that used for the bulk lattice [6] and the details of the implementation in the presence of a wall will be given in a forthcoming paper [4]. The state of column t is a specification of which sites in that column are wet and which are dry and the probability that state i occurs is denoted by $\pi_i(t, p)$. The state in which all sites are dry is labelled $i = 0$. Essentially the state vector of a given column is completely determined by that of the previous column and only one state vector need be held in the computer at any stage. $C(t, x; p)$ is determined by summing $\pi_i(t, p)$ over all states for which the site with coordinate x is wet and

$$r_t(p) = \pi_0(t+1, p) - \pi_0(t, p). \tag{12}$$

Low density expansions in powers of p are obtained by noting that $\pi(t, p) = \mathcal{O}(p^t)$ so that all of the above functions may be obtained to this order by computing the state vectors up to column t. We were able to derive the series directly up to a maximal column $t_m = 49$. However, these series can be extended significantly via an extrapolation method similar to that of [7]. As an example, consider the series for the average cluster length $T(p)$. For each $t < t_m$ we calculate the polynomials $T_t(p) = \sum_{t'=0}^{t} t' r_{t'}(p)$ correct to $\mathcal{O}(p^{70})$. As already noted these polynomials agree with the series for $T(p)$ to $\mathcal{O}(p^t)$. Next, we look at the sequences $d_{t,s}$ obtained from the difference between successive polynomials

$$T_{t+1}(p) - T_t(p) = p^{t+1} \sum_{s \geqslant 0} d_{t,s} p^s. \tag{13}$$

The first of these correction terms $d_{t,0}$ is often a simple sequence which one can readily identify. In this case we find the sequence

$$-d_{t,0} = 1, 2, 3, 6, 10, 20, 35, 70, 126, 252, 462, 924, \ldots$$

from which we conjecture

$$d_{2t,0} = 2d_{2t-1,0} \qquad d_{2t-1} = 4^t / B(t+1, -1/2) \tag{14}$$

where $B(x, y)$ is the beta function. The formula for $d_{t,0}$ holds for all the $t_m - 1$ values that we calculated and we are very confident that it is correct for all values of t. As was the case in [7] the higher-order correction terms $d_{t,s}$ can be expressed as rational functions of $d_{t,0}$,

$$d_{t,s} = \sum_{k=1}^{[s/2]} \binom{t-s}{k} (a_{s,k} d_{t-s+1,0} + b_{s,k} d_{t-s+2,0}) + \sum_{k=0}^{s} c_{s,k} d_{t-s+k+1,0}. \tag{15}$$

From this equation we were able to find formulae for all correction terms up to $s = 17$ and using $T_{49}(p)$ we could extend the series for $T(p)$ to $\mathcal{O}(p^{67})$. A similar procedure allowed us to extend the series for $S(p)$ and the parallel moments $\mu_{1,0}(p)$ and $\mu_{2,0}(p)$ to $\mathcal{O}(p^{67})$, while the series for the first and second perpendicular moments, $\mu_{0,1}(p)$ and $\mu_{0,2}(p)$, were extended to $\mathcal{O}(p^{65})$. The resulting series are listed in table 1. More details of the extrapolation procedure including the formulae for the various correction terms will appear in a later paper [4].

The only high density expansion we consider is that for the percolation probability which can be obtained from (12) and (3) by noting that $r_t(p) = \mathrm{O}(q^k)$ where $q = 1 - p$ and k is the least integer $\geqslant \frac{1}{2}(t+2)$. Thus for a given value of t the number of terms obtainable

Table 1. Low density expansions in powers of p, row n is the coefficient of p^n.

n	$T(p)$	$S(p)$	$\mu_{1,0}(p)$	$\mu_{2,0}(p)$	$\mu_{0,1}(p)$	$\mu_{0,2}(p)$
0	0	1	0	0	0	0
1	1	1	1	1	1	1
2	2	2	4	8	2	4
3	2	3	9	27	5	11
4	5	6	24	96	10	28
5	5	9	47	241	21	65
6	11	17	108	672	40	144
7	13	26	201	1 499	77	303
8	28	47	424	3 676	142	624
9	25	72	762	7 644	262	1 240
10	75	129	1 538	17 398	470	2 438
11	56	194	2 675	34 369	843	4 661
12	188	348	5 258	74 512	1 486	8 872
13	112	516	8 915	141 615	2 609	16 487
14	458	929	17 233	296 939	4 529	30 635
15	319	1 351	28 518	546 394	7 846	55 734
16	1 157	2 456	54 636	1 119 562	13 448	101 618
17	312	3 506	88 459	2 004 015	23 027	181 751
18	3 389	6 471	169 004	4 043 156	39 096	326 608
19	562	8 929	266 670	7 047 626	66 320	575 790
20	9 193	17 029	512 651	14 102 481	111 795	1 022 909
21	−2 419	22 579	786 932	23 956 166	187 946	1 781 314
22	24 689	44 707	1 530 464	47 809 422	314 844	3 135 130
23	−6 090	55 969	2 270 857	79 011 279	526 367	5 402 999
24	83 997	117 836	4 516 598	158 359 672	876 362	9 435 440
25	−80 845	137 313	6 439 085	254 037 643	1 455 579	16 106 911
26	219 791	311 654	13 207 919	514 524 887	2 415 059	27 970 523
27	−95 543	324 989	17 852 082	796 972 392	3 989 542	47 305 236
28	653 560	833 496	38 438 680	1 646 320 650	6 597 538	81 807 186
29	−1 015 961	756 309	48 640 815	2 447 308 375	10 834 513	137 158 135
30	2 302 634	2 242 031	111 440 275	5 201 705 453	17 869 253	236 510 661
31	−2 111 933	1 623 709	128 688 532	7 341 847 456	29 239 356	393 079 288
32	6 978 051	6 176 873	324 010 503	16 294 292 667	48 152 477	677 071 243
33	−12 164 131	3 240 757	331 752 781	21 552 447 211	78 162 313	1 114 451 899

Table 2. Continued

n	$T(p)$	$S(p)$	$\mu_{1,0}(p)$	$\mu_{2,0}(p)$	$\mu_{0,1}(p)$	$\mu_{0,2}(p)$
34	21 361 373	17 192 674	944 134 956	50 707 490 638	128 852 132	1 921 593 186
35	−27 110 387	4 663 165	810 982 473	61 539 314 001	208 370 375	3 130 415 149
36	93 655 507	49 481 888	2 781 591 612	157 488 162 524	343 409 668	5 411 807 564
37	−182 370 254	1 180 046	1 866 117 373	170 712 205 993	549 693 819	8 710 776 761
38	229 034 090	144 593 684	8 270 004 945	489 038 638 889	911 531 157	15 152 834 441
39	−269 557 768	−40 561 669	3 647 454 015	452 466 460 859	1 447 853 041	24 030 119 951
40	1 056 409 556	439 929 287	25 083 883 563	1 526 926 232 817	2 413 312 231	42 187 579 545
41	−2 269 021 879	−230 303 695	5 007 776 568	1 132 548 161 360	3 773 060 280	65 731 749 816
42	2 677 408 443	1 351 358 555	77 130 163 183	4 798 086 858 971	6 361 278 369	117 017 657 827
43	−3 544 761 784	−1 116 634 980	−6 211 741 855	2 514 662 834 523	983 352 727	178 182 324 707
44	13 082 866 127	4 353 263 697	244 028 578 766	15 284 660 803 552	16 833 476 130	323 726 387 136
45	−26 806 541 805	−4 398 416 071	−83 631 438 989	4 380 744 364 749	25 157 427 559	478 236 033 969
46	26 061 243 131	14 001 291 871	783 204 867 296	49 292 061 993 412	44 287 084 338	894 531 996 536
47	−40 361 968 343	−17 738 446 374	−494 314 396 278	989 931 047 506	64 933 486 366	1 270 849 732 090
48	190 465 471 378	47 119 949 250	2 594 611 285 466	162 241 456 668 132	117 606 789 796	2 471 975 021 852
49	−381 128 060 099	−64 270 709 097	−2 232 294 549 879	−39 805 018 765 919	161 582 598 415	3 328 670 679 553
50	225 643 036 457	157 128 098 347	8 690 778 026 386	542 342 994 602 556	311 756 741 490	6 859 481 787 132
51	−287 003 337 097	−246 380 178 827	−9 661 864 892 692	−284 699 866 038 824	408 491 249 744	8 579 303 387 168
52	2 566 759 769 655	545 460 020 544	29 995 760 431 218	1 856 106 540 303 732	841 943 528 892	19 102 460 884 304
53	−5 285 267 101 147	−862 856 345 434	−38 056 677 957 915	−1 431 588 334 552 263	968 313 512 109	21 611 403 485 081
54	2 271 123 259 017	1 858 869 421 298	103 906 790 631 563	6 441 871 877 547 593	2 256 308 657 115	53 709 860 916 525
55	−3 165 468 030 218	−3 252 844 644 627	−151 969 740 070 893	−6 562 243 329 132 823	2 354 715 740 977	52 606 208 892 861
56	35 212 809 299 763	6 592 890 548 347	369 827 081 677 281	22 869 643 990 253 339	6 364 532 607 737	152 781 299 898 183
57	−66 427 001 953 763	−11 229 139 704 329	−570 503 946 433 867	−27 580 998 453 503 811	4 823 911 367 581	121 017 115 594 937
58	11 057 548 952 493	22 767 401 371 634	1 310 843 427 572 251	819 223 44 320 438 959	17 432 800 454 267	441 260 107 224 351
59	−31 697 059 334 297	−42 147 789 558 521	−2 209 141 231 427 900	−114 301 635 466 580 028	10 767 177 749 158	253 668 652 604 268
60	531 845 697 600 814	81 707 816 765 666	4 757 125 831 653 685	299 099 704 878 008 319	52 298 853 703 005	1 298 380 307 866 003
61	−93 9850 501 378 691	−144 224 611 556 818	−8 109 804 036 235 413	−452 153 132 335 049 221	102 740 67 757 479	411 221 700 812 127
62	−218 089 303 232 488	284 988 594 853 047	17 109 904 775 959 109	1 095 748 251 643 358 129	149 825 804 840 191	3 920 538 018 919 121
63	146 310 515 780 374	−544 069 973 568 349	−31 055 984 288 473 750	−1 802 157 080 659 641 406	3 194 083 769 764	164 257 826 455 782
64	8 010 088 501 049 393	1 029 622 326 675 184	62 805 743 084 099 736	4 074 933 118 400 663 972	488 096 955 080 292	12 077 039 640 386 216
65	−13 777 249 481 066 198	−1 844 661 752 754 855	−112 541 611 208 180 874	−6 931 655 775 629 313 164	−219 315 581 678 014	−3 036 358 866 297 604
66	−7 335 657 891 417 937	3 612 493 459 852 700	227 780 508 663 102 551	15 135 810 090 250 397 585		
67	5 810 530 478 862 470	−7 025 211 744 800 954	−429 949 623 442 589 455	−27 153 914 600 589 832 779		

in the high density expansion is only about half as many as in the low density expansion. However, for computational purposes it is more efficient to derive the series expansion for $P(q)$ directly via a transfer matrix technique. For the percolation probability we derived the series directly to $\mathcal{O}(q^{24})$ and obtained another eight terms from the extrapolation procedure. The resulting series is listed in table 2.

Table 2. High density expansion for the percolation probability $P(q) = \sum a_n q^n$.

n	a_n	n	a_n
0	1	17	−123 721
1	−1	18	−287 828
2	−2	19	−790 641
3	−3	20	−1 875 547
4	−4	21	−5 302 725
5	−7	22	−12 258 340
6	−11	23	−35 837 868
7	−24	24	−83 642 760
8	−44	25	−242 399 471
9	−108	26	−569 416 045
10	−221	27	−1 704 989 414
11	−563	28	−3 898 028 574
12	−1 234	29	−11 682 423 741
13	−3 240	30	−28 476 236 374
14	−7 221	31	−80 448 369 426
15	−19 835	32	−194 172 723 271
16	−44 419		

It is found from unbiased approximants that the estimates of p_c agree with the bulk value [6], $p_c = 0.644\,7002 \pm 0.000\,0005$ obtained from longer series and we therefore bias our exponent estimates with this value. This value of p_c was obtained from low density series and is a refinement of that obtained from analysis of the shorter series for $P(q)$ [8] which gave $p_c = 0.644\,7006 \pm 0.000\,0010$. Data obtained from $T(p)$, the parallel moments and $P(q)$ are shown in tables 3, 4 and 5. The exponent of $\mu_{00}(p)$ was estimated from the series for $(S(p)-1)/p$ which is the mean size of the cluster connected to the site (1, 1); this gave better convergence. We have also analysed the first and second perpendicular moment of the pair connectedness and series for $\xi_{\parallel}(p)$ and $\xi_{\perp}(p)$ obtained from (11) using the first and second moments. In the analysis of $P(q)$ we used standard DLog Padé approximants while the remaining series were analysed using first- and second-order inhomogeneous differential approximants [9].

In table 3 the columns headed $L = 0$ result from the standard DLog Padé analysis and give $\tau_1 = 1$ to three decimal places although most of the entries are slightly above. This conclusion is not altered by looking at inhomogeneous approximants (the first few of which we have included in table 3) or second-order approximants. Using the slightly smaller value $p_c = 0.644\,6980$ gave the better converged result $\tau_1 = 1.000\,04 \pm 0.000\,04$.

We turn now to the indirect evidence for $\tau_1 = 1$ via the scaling relation (1). The value $\nu_{1\parallel} = 1.7337 \pm 0.0004$ was obtained by analysing the series for $\mu_{2,0}(p)/\mu_{1,0}(p)$ and is consistent with the value obtained by subtracting the value of the exponent of $\mu_{1,0}$ from that of $\mu_{2,0}$. It is clearly equal to the corresponding bulk exponent, as in the case of compact percolation, and we use the more accurate bulk estimate in deriving τ_1 below. The corrections to scaling in the case of the percolation probability appear to be very close to analytic, and the standard Padé estimate of β_1 (table 6) should be accurate. Combining the

Table 3. Differential approximant analysis of the mean length series. The table shows biased first-order inhomogeneous approximant estimates of τ_1. L is the degree of the inhomogeneous polynomial. For $L = 0$ the entries are from biased Dlog Padé approximants.

	$L = 0$			$L = 1$		
N	$[N-1, N]$	$[N, N]$	$[N+1, N]$	$[N-1, N]$	$[N, N]$	$[N+1, N]$
22	1.000 10	1.000 10	1.000 10	1.000 10	1.000 14	1.000 10
23	1.000 10	1.000 10	1.000 09	1.000 07	0.999 23	1.000 12
24	1.000 10	1.000 15	1.000 19	1.000 12	1.000 06	1.000 15
25	1.000 19	1.000 08	1.000 21	1.000 15	1.000 16	1.000 19
26	1.000 21	1.000 20	1.000 21	1.000 19	1.000 20	1.000 06
27	1.000 21	1.000 12	1.000 27	1.000 10	1.000 25	1.000 33
28	1.000 28	1.000 24	1.000 34	1.000 36	1.000 23	1.000 72
29	1.000 01	1.000 20	1.000 24	1.000 10	1.000 20	1.000 21
30	1.000 28	1.000 22	1.000 22	1.000 21	1.000 22	1.000 26
31	1.000 22	1.000 29	1.000 23	1.000 26	1.000 25	1.000 23
32	1.000 23	1.000 22	1.000 22	1.000 23	1.000 22	1.000 22
33	1.000 22	1.000 22		1.000 22		
	$L = 0$			$L = 1$		
N	$[N-1, N]$	$[N, N]$	$[N+1, N]$	$[N-1, N]$	$[N, N]$	$[N+1, N]$
22	1.000 10	1.000 01	0.999 54	1.000 10	0.999 96	1.000 14
23	0.999 62	1.000 04	1.000 09	1.000 14	1.000 08	1.000 11
24	1.000 09	0.999 96	1.000 14	1.000 12	1.000 17	1.000 15
25	1.000 14	1.000 17	1.001 44	1.000 15	1.000 19	1.000 34
26	0.999 86	1.000 28	1.000 30	1.000 38	1.000 30	1.000 29
27	1.000 30	1.000 8	0.999 95	1.000 30	1.000 29	1.000 18
28	1.000 14	1.000 20	1.000 20	1.000 19	1.000 20	1.000 22
29	1.000 20	1.000 20	1.000 25	1.000 23	1.000 22	1.000 22
30	1.000 30	1.000 22	1.000 23	1.000 22	1.000 23	1.000 23
31	1.000 23	1.000 23	1.000 22	1.000 23	1.000 22	1.000 23
32	1.000 23	1.000 23		1.000 23		

values of $\nu_\parallel$ and β_1 gives $\tau_1 = 1.0000 \pm 0.0002$ which agrees with the direct estimate.

Other exponent values obtained from the analysis of various series are collected together in table 6 where previous estimates for the bulk problem and exact results for compact percolation are also given. As usual the error bars are a measure of the consistency of the higher-order approximants and are not strict bounds. The estimate $\beta = 0.276\,43 \pm 0.000\,10$ of [8] has been adjusted slightly upwards to allow for the change in p_c. In estimating the exponents we rely both on the analysis of the series yielding a particular exponent and estimates obtained using scaling relations. In some cases we also use the more accurate bulk exponent estimates. A case in point is the exponent γ_1. From the Dlog Padé approximants in table 4 one would say that the direct estimate from the series for $(S(p) - 1)/p$ favours a value of $\gamma_1 \simeq 1.8211$ with a rather large spread among the approximants. However, the better converged estimates of $\gamma_1 + 2\nu_{1\parallel} \simeq 5.2881$ together with the bulk estimate of $\nu_\parallel$ leads to $\gamma_1 \simeq 1.8205$. In this case second-order differential approximants to $S(p)$ are better converged and favour $\gamma_1 \simeq 1.8207$. Taking all the evidence into account including our belief that Δ_1 takes on the bulk value we arrived at the estimate for γ_1 quoted in table 6. The estimate of τ is derived from the scaling relation (9). Analysis of the bulk expansions [6, 8, 10] showed that corrections to scaling were close to analytic, as they are here.

The values of $\nu_{1\perp}$ and Δ_1 (obtained from the scaling relation (2)), as well as $\nu_{1\parallel}$, are

Table 4. DLog Padé analysis of the moments of the pair connectedness. The table shows biased approximant estimates of the critical exponents of the moments $\mu_{00}(p)$, $\mu_{10}(p)$ and $\mu_{20}(p)$.

	γ_1			$\gamma_1 + \nu_{1\parallel}$			$\gamma_1 + 2\nu_{1\parallel}$		
N	$[N-1, N]$	$[N, N]$	$[N+1, N]$	$[N-1, N]$	$[N, N]$	$[N+1, N]$	$[N-1, N]$	$[N, N]$	$[N+1, N]$
22	1.823 81	1.827 60	1.819 53	3.554 92	3.555 55	3.554 58	5.288 07	5.288 07	5.288 08
23	1.820 10	1.825 93	1.823 55	3.554 66	3.554 78	3.554 72	5.288 09	5.287 69	5.288 07
24	1.823 64	1.820 94	1.717 66	3.554 73	3.554 79	3.554 62	5.288 08	5.288 04	5.288 09
25	1.774 37	1.815 58	1.823 99	3.554 66	3.554 56	3.554 57	5.288 09	5.288 09	5.288 09
26	1.825 11	1.827 93	1.824 24	3.554 57	3.554 56	3.554 59	5.288 05	5.288 09	5.288 09
27	1.825 24	1.820 63	1.821 22	3.554 60	3.554 59	3.554 59	5.288 09	5.288 08	5.288 06
28	1.821 24	1.820 97	1.820 78	3.554 60	3.554 59	3.554 59	5.288 07	5.288 09	5.285 16
29	1.820 79	1.820 90	1.823 96	3.554 60	3.554 53	3.554 58	5.288 04	5.288 02	5.288 19
30	1.818 78	1.820 98	1.821 07	3.554 59	3.554 58	3.554 59	5.288 04	5.288 06	5.287 99
31	1.821 08	1.821 08	1.821 07	3.554 3	3.554 54	3.554 63	5.288 05	5.287 61	5.287 99
32	1.821 08	1.821 04	1.821 06	3.554 25	3.554 71	3.554 60	5.288 05	5.288 06	5.288 08
33	1.821 06	1.821 04		3.555 89	3.554 71		5.288 03	5.288 06	

Table 5. DLog Padé analysis of the percolation probability series. The table shows biased approximant estimates of β_1.

N	$[N-1, N]$	$[N, N]$	$[N+1, N]$
8	0.734 06	0.734 06	0.734 08
9	0.734 09	0.734 09	0.734 08
10	0.734 09	0.734 03	0.733 69
11	0.733 89	0.733 88	0.733 85
12	0.733 89	0.733 81	0.733 82
13	0.733 82	0.733 81	0.733 82
14	0.733 83	0.733 81	0.733 82
15	0.733 82	0.733 82	0.733 79
16	0.733 80	0.733 82	

clearly the same as those for the bulk. The scaling size and both scaling lengths are therefore unchanged by the introduction of the wall. We also note that the hyperscaling relation, with D the dimension of space perpendicular to the preferred direction t ($= 1$ for the square lattice),

$$\nu_\parallel + D\nu_\perp = \beta + \Delta \tag{16}$$

which is satisfied by the bulk exponents apparently fails on the introduction of a wall.

We now consider the possibility of rational exponents. As previously noted [8], there is no simple rational fraction whose decimal expansion agrees with the estimate of β. The same is true for other exponent estimates in table 6. In particular we note that our estimates of the bulk exponents $\nu_{||}$ and $\nu_\perp$ differ by 0.03% from the rational fractions $\nu_{||} = 26/15 = 1.733\,333\ldots$, and $\nu_\perp = 79/72 = 1.097\,222\ldots$ suggested by Essam *et al* [11]. We believe this to be a significant difference given the high precision of our results. However, the suggested rational fraction $\gamma = 41/18 = 2.277\,777\ldots$ and the value of $\Delta = 613/240 = 2.554\,1666\ldots$, which follows from the above rational values by scaling, are generally still within our estimated the error bounds. The fraction for Δ is not very appealing though and assuming that both exponents have these values then scaling implies the even less convincing result $\beta = 199/720 = 0.276\,388\ldots$ which is however just

Table 6. Exponent values for compact and bond percolation. The bulk values for bond percolation are from [6] except for β which is from [8], adjusted for a small change in p_c. The compact percolation results are from [2] and references therein. Values in brackets are obtained from scaling formulae. The 'with wall' value of γ is from second-order differential approximants.

	Bond percolation		Compact percolation	
Exponent	With wall	Bulk	With wall	Bulk
τ	1.0002 ± 0.0003	(1.4573 ± 0.0002)	0	1
β	0.7338 ± 0.0001	0.27647 ± 0.00010	2	1
γ	1.8207 ± 0.0004	2.2777 ± 0.0001	1	2
$\gamma + \nu_{\parallel}$	3.5546 ± 0.0002	4.0113 ± 0.0003	(3)	(4)
$\gamma + 2\nu_{\parallel}$	5.2881 ± 0.0002	5.7453 ± 0.0004	(5)	(6)
$\nu_{\parallel}$	1.7337 ± 0.0004	1.7338 ± 0.0001	(2)	2
$\gamma + 2\nu_{\perp}$	4.014 ± 0.002	4.4714 ± 0.0004		(3)
$\nu_{\perp}$	1.0968 ± 0.0003	1.0969 ± 0.0001		1
Δ	(2.5545 ± 0.0005)	(2.5542 ± 0.0002)	3	3

Table 7. Scaling values of the exponents for bond percolation calculated using $\tau_1 = 1$, $\gamma = \frac{41}{18}$ and the bulk estimates of $\nu_{\parallel}$ and $\nu_{\perp}$.

Exponent	With wall	Bulk
τ	1	1.4573
β	0.7338	0.27646
γ	1.8204	2.2778
$\gamma + \nu_{\parallel}$	3.5542	4.0116
$\gamma + 2\nu_{\parallel}$	5.2880	5.7454
$\nu_{\parallel}$	1.7338	1.7338
$\gamma + 2\nu_{\perp}$	4.0142	4.4716
$\nu_{\perp}$	1.0969	1.0969
Δ	2.5542	2.5542

consistent with our estimated value.

If we assume that $\tau_1 = 1$ is exact and that the values of Δ, $\nu_{\parallel}$ and $\nu_{\perp}$ are the same with and without a wall then all of the other surface exponents are determined by scaling together with the values of any three bulk exponents. The surface exponents calculated in this way are presented in table 7 for comparison with the estimated values of table 6 as a measure of the overall consistency of our results. The bulk exponents used were $\gamma = 41/18$ and the bulk estimates of $\nu_{||}$ and $\nu_{\perp}$. Excellent agreement is observed.

Our findings may be summarized as follows. Firstly we have found that the scaling size and both scaling length exponents are unchanged by the introduction of a wall parallel to the preferred direction. Also we have examined the widely held view that two-dimensional systems should have rational exponents. The high precision data presented here are consistent with the results $\tau_1 = 1$ and $\gamma = 41/18$. However there are no such simple fractions which are in agreement with our estimates of $\nu_{||}$ and $\nu_{\perp}$. Given that directed percolation is not conformally invariant, and that the expectation of exponent rationality is a consequence of conformal invariance, this is perhaps not surprising. The precise numerical work reported and quoted in this paper therefore supports the conclusion that the critical exponents for non-translationally invariant models should not, in general, be expected to be simple rational numbers. The cluster length exponent τ_1 and the exponent γ appear to be exceptional cases.

References

[1] Essam J W and TanlaKishani D 1994 *J. Phys. A: Math. Gen.* **27** 3743
[2] Essam J W and Guttmann A J 1995 *J. Phys. A: Math. Gen.* **28** 3591
[3] Lin J-C 1992 *Phys. Rev.* A **45** R3394
[4] Jensen I and Guttmann A J in preparation
[5] Stauffer D 1985 *Introduction to Percolation Theory* (London: Taylor & Francis)
[6] Jensen I in preparation
[7] Baxter R J and Guttmann A J 1988 *J. Phys. A: Math. Gen.* **21** 3193
[8] Jensen I and Guttmann A J 1995 *J. Phys. A: Math. Gen.* **28** 4813
[9] Guttmann A J 1989 Asymptotic analysis of power-series expansions *Phase Transitions and Critical Phenomena* vol 13, ed C Domb and J Lebowitz (New York: Academic) pp 1–234
[10] Essam J W, De'Bell K, Adler J and Bhatti F M 1986 *Phys. Rev.* B **33** 1982
[11] Essam J W, Guttmann A J and De'Bell K 1988 *J. Phys. A: Math. Gen.* **21** 3815